Decomposition Theory of Spin Connection and Topological Structure of Gauss-Bonnet-Chern Theorem on Manifold With Boundary

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Abstract

The index theorem of Euler-Poincaré characteristic of manifold with boundary is given by making use of the general decomposition theory of spin connection. We shows the sum of the total index of a vector field ϕ and half the total of the projective vector field of ϕ on the boundary equals the Euler-Poincaré characteristic of the manifold. Detailed discussion on the topological structure of the Gauss-Bonnet-Chern theorem on manifold with boundary is given. The Hopf indices and Brouwer degrees label the local structure of the Euler density.

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I. INTRODUCTION

Gauss-Bonnet-Chern(GBC) theorem is one of the most significant results in differential geometry, which relates the curvature of the compact and oriented even-dimensional Riemannian manifold $\mathbf{M}$ with an important topological invariant, the Euler-Poincaré characteristic $\chi(\mathbf{M})$. An elegantly intrinsic proof of the theorem was given by professor Chern [15], whose instructive idea is to work on the sphere bundle $S(\mathbf{M})$. An useful summary and some historical comments on the GBC theorem are given respectively by Kabayashi, Nomizu [1], and Spivak [2]. A detailed review of Chern's proof of the GBC theorem is presented in Ref. [3].

A great advance in this field was the discovery of the relationship between supersymmetry and the index theorem, which includes the derivation of the GBC theorem via supersymmetry and the path integral techniques as presented by Alvarez-Gaumé et al [4]. In topological quantum field theory which was initiated by Witten [5], the GBC theorem can be derived by means of Morse theory [6]. On the physics side, the optical Berry phase is a direct result of the Gauss-Bonnet theorem [7] and the black hole entropy emerges as the Euler class through dimensional continuation of the Gauss-Bonnet theorem [8].

In this paper a generalized Hopf theorem on manifolds with boundaries is given. It says the Euler-Poincaré characteristic χ of a manifold M with boundary ∂M equals the sum of the total index the vector field ϕ on M and half the total index of the projection vector of ϕ on the boundary ∂M. A detailed discussion on the local structure of the GBC density (the Euler-Poincaré characteristic $\chi(\mathbf{M})$ density) shows that only the zeroes of ϕ and the zeroes of the projective vector of ϕ on ∂M contribute to $\chi(\mathbf{M})$. We also show that the Brouwer degrees and Hopf indices label the local structure of the GBC density. The direct relationship between the Euler-Poincaré characteristic and winding number is given.

The generalized decomposition theory of spin connection using in this paper is an useful tool in the research of topology in gauge theory. This theory has been effectively used in the magnetic monopole problem [9,10], the topological gauge theory of dislocation and disclination in condensed matter physics [11], the space-time defects [12] and Gauss-Bonnet-Chern(GBC) theorem [13,14]. The essential feature of the decomposition theory shows that the connection can be decomposed and possess inner structures. In this paper, we give a general decomposition and inner structure of the spin connection of the group $SO(N)$ using N orthonormal vectors via Clifford algebra. The curvature under this decomposition is found to be a generalized function. The singular points of the curvature are the sources of the non-triviality of the manifold.

This paper is arranged as follows. In section 2, we deduce the general decomposition and inner structure of spin connection via N orthonormal vectors. Using the formula given in section2, the GBC form is expressed in terms of an unit vector in section 3, In section 4 and 5, the topological structure of the Chern density on manifold without boundary and without boundary are given respectively. In section 6, we present a short summary.

II. THE DECOMPOSITION THEORY OF SPIN CONNECTION

Let $\mathbf{M}$ be a compact and oriented N-dimensional Riemannian manifold and $P(\pi, \mathbf{M}, G)$ be a principal bundle with the structure group $G = SO(N)$. A smooth vector field ϕ^a ($a = 1, 2, \cdots, N$) can be found on the base manifold $\mathbf{M}$ (a section of a vector bundle over $\mathbf{M}$). We define a unit vector n on $\mathbf{M}$ as

$$n^a = \phi^a / ||\phi|| \qquad a = 1, 2, \cdots, N \tag{1}$$

$$||\phi|| = \sqrt{\phi^a \phi^a},$$

in which the superscript "a" is the local orthonormal frame index.

In fact n is identified as a section of the sphere bundle over $\mathbf{M}$ (or a partial section of the vector bundle over $\mathbf{M}$). We see that the zeroes of ϕ are just the singular points of n. Since the global property of a manifold has close relation with zeroes of a smooth vector fields on it, this expression of the unit vector $\vec{n}$ is a very powerful tool in the discussion of the global topology. It naturally guarantees the constraint

$$n^a n^a = 1, \tag{2}$$

and

$$n^a dn^a = 0. \tag{3}$$

The covariant derivative 1-form of n^a is

$$Dn^a = dn^a - \omega^{ab} n^b, \tag{4}$$

and the curvature 2-form is

$$F^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{cb}, \tag{5}$$

where ω^{ab} is the spin connection 1-form:

$$\omega^{ab} = \omega^{ab}_{\mu} dx^{\mu} \qquad \omega^{ab} = -\omega^{ba}, \tag{6}$$

and

$$F^{ab} = \frac{1}{2} F^{ab}_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \qquad F^{ab} = -F^{ba}. \tag{7}$$

Using (3), (4) and (6) it can be shown that

$$n^a Dn^a = 0. \tag{8}$$

Let the N-dimensional Dirac matrix γ_a ($a = 1, 2, \cdots, N$) be the basis of the Clifford algebra which satisfies

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab}. \tag{9}$$

An unit vector field $\vec{n}$ on $\mathbf{M}$ can be expressed as a vector of Clifford Algebra

$$n = n^a \gamma_a, \tag{10}$$

which can be correspondingly written as

$$n = \frac{1}{||\phi||}\phi, \tag{11}$$

where ϕ is a Clifford Algebra vector also

$$\phi = \phi^a \gamma_{a.} \tag{12}$$

The spin connection 1-form and curvature 2-form are respectively represented as Clifford-algebra-valued differential forms

$$\omega = \frac{1}{2}\omega^{ab} I_{ab} \qquad F = \frac{1}{2}F^{ab} I_{ab}, \tag{13}$$

in which I_{ab} is the generator of the spin representations of the group $SO(N)$

$$I_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] = \frac{1}{4}(\gamma_a\gamma_b - \gamma_b\gamma_a). \tag{14}$$

By (4) and making use of (9) and (10), it's easy to prove that the covariant derivative 1-form of n^a can be represented in terms of n and ω

$$Dn = dn - [\omega, n], \tag{15}$$

and curvature 2-form

$$F = d\omega - \omega \wedge \omega. \tag{16}$$

Arbitrary $U \in Spin(N)$, which satisfies

$$UU^\dagger = U^\dagger U = I, \tag{17}$$

is an even-versor [17]. The induced 'spinorial' transformation by U to the basis γ_i of the Clifford algebra give N orthonormal vectors u_i [18] via

$$u_i := U\gamma_i U^\dagger = u_i^a \gamma_a, \tag{18}$$

where u_i^a is the coefficient of u_i in the representation of Clifford algebra. From the relationship between U and u_i^a, we see that u_i has the same singular points with respect to different "i". By (18), it is easy to verify that u_i satisfy

$$u_i u_j + u_j u_i = 2\delta_{ij}, \qquad i, j = 1, 2, \cdots N. \tag{19}$$

From (15) we know that the covariant derivative 1-form of u_i is

$$Du_i = du_i - [\omega, u_i]. \tag{20}$$

There exists the following formula for a Clifford Algebra r-vector A [17]

$$u_i A u_i = (-1)^r (N - 2r) A. \tag{21}$$

For ω is a Clifford Algebra 2-vector and using (21), the spin connection ω can be decomposed by N orthonormal vectors u_i as

$$\omega = \frac{1}{4}(du_i u_i - Du_i u_i). \tag{22}$$

It can be proved that the general decomposition formula (22) has global property and is independent of the choice of the local coordinates [14].

By choosing the gauge condition

$$Du_i = 0 \tag{23}$$

we can define a generalized pseudo-flat spin connection as

$$\omega_0 = \frac{1}{4} du_i u_i. \tag{24}$$

Suppose there exist l singular points z_i $(i = 1, 2, \cdots, l.)$ in the orthonormal vectors u_j. One can easily prove that at the normal points of u_j

$$F(\omega_0) = 0 \qquad when \quad x \neq z_i. \tag{25}$$

For the derivative of u_i at the singular points z_i is undefined, the formula (25) is invalid at z_i.

$$F \begin{cases} = 0 \\ \neq 0 \end{cases} \qquad when \quad \begin{matrix} x \neq z_i, \\ x = z_i. \end{matrix} . \tag{26}$$

This is why we call ω_0 the pseudo-flat spin connection.

III. THE GAUSS-BONNET-CHERN FORM

The GBC form is a N(even)-form Λ over an even dimensional compact and oriented Riemannian manifold $\mathbf{M}$, such that when pulled back to $S(\mathbf{M})$, it's exact

$$\pi^* \Lambda = d\Omega, \tag{27}$$

where

$$\Lambda = \frac{1}{\pi^{N/2} N!!} Tr(\gamma F \wedge F \cdots \wedge F). \tag{28}$$

Using a recursion method Chern [15] has proved that the $(N-1)$ form Ω on $S(\mathbf{M})$ is

$$\Omega = \frac{1}{(2\pi)^{N/2}} \sum_{k=0}^{N/2-1} (-1)^k \frac{2^{-k}}{(N-2k-1)!!k!} \Theta_k, \qquad N \geq 4 \tag{29}$$

which is called the Chern form with

$$(N-2k-1)!! = (N-2k-1)(N-2k-3)(N-2k-5)\cdots 1, \tag{30}$$

$$\Theta_k = \frac{(-1)^{N/2}}{2^{(N-2k)/2}} Tr(\gamma n Dn \wedge \cdots \wedge Dn \wedge F \wedge \cdots \wedge F).$$

By virtue of the Bianchi identity

$$DF = 0, \tag{31}$$

it can be proved that

$$D\Lambda = 0. \tag{32}$$

Thus Λ determines a cohomology class belonging to the cohomology groups $H^N(\mathbf{M})$ of $\mathbf{M}$, which is independent of the connection. This is equivalent to saying that the integral $\int_{\mathbf{M}}\Lambda$ taken over a closed manifold $\mathbf{M}$ is a topological invariant [20]: the Euler-Poincaré characteristic $\chi(\mathbf{M})$. We note that π^* maps the cohomology of $\mathbf{M}$ into that of $S(\mathbf{M})$, while n^* performs the inverse operation, i.e., $n^*\pi^*$ amounts to the identity. The famous GBC theorem can thus be expressed as

$$\chi(\mathbf{M}) = \int_{\mathbf{M}} \Lambda = \int_{\mathbf{M}} n^*\pi^*\Lambda = \int_{\mathbf{M}} n^* d\Omega. \tag{33}$$

Pull back to $S(\mathbf{M})$, (33) becomes

$$\chi(\mathbf{M}) = \int_{n(\mathbf{M})} d\Omega. \tag{34}$$

When the manifold $\mathbf{M}$ has boundary, i.e. $\partial\mathbf{M} \neq 0$, Gilkey [23] showed that

$$\chi(\mathbf{M}) = \int_{n(\mathbf{M})} d\Omega + \int_{n(\partial\mathbf{M})} \Omega \tag{35}$$

From the well-known Chern-Weil homomorphism [1,24], we know that the Euler classes is independent of the connection. Hence, we have many choices of spin connection and the choice depends on the convenience of calculus. In the present research, we use the generalized pseudo-flat spin connection to compute the Euler number.

Let ω takes as the pseudo-flat spin connection (24). Then the curvature 2-form vanishes everywhere but the singular points z_l of u_i. It is clearly to see

$$\Omega = \frac{(-1)^{N/2}}{2^N \pi^{N/2}(N-1)!!} Tr(\gamma n Dn \wedge Dn \wedge \cdots \wedge Dn) \tag{36}$$

According to (21), n can be written as

$$n = n_i u_i, \tag{37}$$

where n_i is the projection of n on u_i

$$n_i = \frac{1}{2}(nu_i + u_i n). \tag{38}$$

Noticing that u_i and du_i are both Clifford Algebra vectors, using (21) we have

$$u_j u_i u_j = -(N-2)u_i, \qquad u_j du_i u_j = -(N-2)du_i. \tag{39}$$

Then Dn becomes

$$Dn = dn_i u_i. \tag{40}$$

As a result

$$\Omega = \frac{(-1)^{N/2}}{2^N \pi^{N/2}(N-1)!!} Tr(\gamma u_{i_1} u_{i_2} \cdots u_{i_N}) n_{i_1} dn_{i_2} \wedge \cdots \wedge dn_{i_N}. \tag{41}$$

It can be deduced that Ω is

$$\Omega = \frac{1}{(n-1)! A(S^{N-1})} \epsilon_{a_1 a_2 \cdots a_N} n_{a_1} dn_{a_2} \wedge \cdots \wedge dn_{a_N}. \tag{42}$$

where $A(S^{N-1})$ is the area of S^{N-1}

$$A(S^{N-1}) = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})}. \tag{43}$$

For n_a are just the projections of n on the basis u_a, we have

$$n_a n_a = 1. \tag{44}$$

The above expression (42) is nothing but the GBC-form expressed cleanly in terms of the unit vector n.

IV. TOPOLOGICAL STRUCTURE OF *GBC* DENSITY ON THE MANIFOLD WITHOUT BOUNDARY

We can write n_a as

$$n_a = \frac{1}{2}(nu_a + u_a n) = \frac{\frac{1}{2}(\phi u_a + u_a \phi)}{||\phi||} = \frac{\phi_a}{||\phi||}, \tag{45}$$

where ϕ_a are the projections of ϕ on $u_{(a)}$, so it is easy to prove that

$$||\phi|| = \sqrt{\phi^a \phi^a} = \sqrt{\phi_a \phi_a}. \tag{46}$$

The derivative of n_a can be deduced as

$$dn_a = \frac{d\phi_a}{||\phi||} - \phi_a d(\frac{1}{||\phi||}). \tag{47}$$

Substituting it into (42), we have the expression of Ω on $S(\mathbf{M})$

$$\Omega = \frac{1}{A(S^{N-1})(N-1)!}\epsilon_{a_1a_2\cdots a_N}\frac{\phi_{a_1}}{||\phi||^N}d\phi_{a_2}\wedge\cdots\wedge d\phi_{a_N}. \tag{48}$$

Using

$$\frac{\phi_a}{||\phi||^N} = -\frac{1}{N-2}\frac{\partial}{\partial\phi_a}(\frac{1}{||\phi||^{N-2}}), \tag{49}$$

the pull back of the exterior derivative of (48) to $\mathbf{M}$ can be written as

$$\begin{aligned} n^*d\Omega = &-\frac{1}{A(S^{N-1})(N-1)!(N-2)}\epsilon_{a_1a_2\cdots a_N}\frac{\partial}{\partial\phi_a}\frac{\partial}{\partial\phi_{a_1}}(\frac{1}{||\phi||^{N-2}}) \\ &\times\frac{\partial\phi_a}{\partial x^{\mu_1}}\frac{\partial\phi_{a_1}}{\partial x^{\mu_2}}\cdots\frac{\partial\phi_{a_N}}{\partial x^{\mu_N}}\frac{\epsilon^{\mu_1\mu_2\cdots\mu_N}}{\sqrt{g}}\sqrt{g}d^Nx, \end{aligned} \tag{50}$$

where $g = det(g_{\mu\nu})$, $g_{\mu\nu}$ is the metric tensor of $\mathbf{M}$. Define the Jacobian $D(\phi/x)$ as

$$\epsilon_{a_1a_2\cdots a_N}D(\phi/x) = \epsilon^{\mu_1\mu_2\cdots\mu_N}\partial_{\mu_1}\phi_{a_1}\partial_{\mu_2}\phi_{a_2}\cdots\partial_{\mu_N}\phi_{a_N}. \tag{51}$$

Noticing

$$\epsilon_{a_1a_2\cdots a_N}\epsilon_{aa_2\cdots a_N} = (N-1)!\delta_{a_1a}, \tag{52}$$

we get

$$n^*d\Omega = -\frac{1}{A(S^{N-1})(N-2)}\frac{\partial^2}{\partial\phi_a\partial\phi_a}(\frac{1}{||\phi||^{N-2}})D(\frac{\phi}{x})d^Nx. \tag{53}$$

The general Green's-function formula [21] in ϕ space is

$$\triangle_\phi(\frac{1}{||\phi||^{N-2}}) = -\frac{4\pi^{N/2}}{\Gamma(N/2-1)}\delta(\phi) \qquad N\geq 3, \tag{54}$$

where

$$\triangle_\phi = \frac{\partial^2}{\partial\phi_a\partial\phi_a}, \tag{55}$$

is the N-dimensional Laplacian operator in ϕ space. We obtain the new formulation of GBC form in terms of δ function $\delta(\phi)$

$$n^*d\Omega = \delta(\phi)D(\phi/x)d^Nx. \tag{56}$$

Since $\phi^a(x)$ has l isolated zeroes on $\mathbf{M}$ and let the ith zero be z_i, it is well known from the ordinary theory of the δ-function [22] that

$$\delta(\phi) = \sum_{i=1}^{l}\frac{\beta_i\delta(x-z_i)}{D(\phi/x)|_{x=z_i}}. \tag{57}$$

Then one obtains

$$\delta(\phi)D(\frac{\phi}{x}) = \sum_{i=1}^{l} \beta_i \eta_i \delta(x - z_i), \tag{58}$$

where β_i is the positive integer (the Hopf index of the ith zero) and η_i the Brouwer degree [20,16]

$$\eta_i = sgn D(\phi/x)|_{x=z_i} = \pm 1. \tag{59}$$

From the above deduction the following topological structure is obtained:

$$n^* d\Omega = \delta(\phi)D(\frac{\phi}{x})d^N x = \sum_{i=1}^{l} \beta_i \eta_i \delta(x - z_i) d^N x, \tag{60}$$

which means that the local structure of $n^* d\Omega_0$ is labeled by the Brouwer degrees and Hopf indices, which are topological invariants. Therefore the Euler-Poincaré characteristic $\chi(M)$ can be represented as

$$\chi(\mathbf{M}) = \int_{\mathbf{M}} n^* d\Omega = \sum_{i=1}^{l} \beta_i \eta_i. \tag{61}$$

On another hand the above formula also gives the winding number of the manifold M and the mapping $\vec{\phi}$ (see, for e.g [])

$$W(\phi, z_i) = \beta_i \eta_i.$$

Then the Euler-Poincaré characteristic $\chi(\mathbf{M})$ can further be expressed in terms of winding numbers and degree of ϕ

$$\chi(\mathbf{M}) = \deg \phi = W(\phi, \mathbf{M}) = \sum_{i=1}^{l} W(\phi, z_i). \tag{62}$$

From (1), we know that the zeroes of ϕ are just the singularities of n. Here (61) says that the sum of the indices of the singular points of n, or of the zeroes of ϕ, is the Euler-Poincaré characteristic. Therefore the topological structure of GBC density reveals the expected result of the Hopf theorem. The above discussions, especially, the expressions (60) is very valuable to establish the theory of the GBC topological current. Since the GBC theorem is also correct for a pseudo-Riemannian manifold [19], from (60) we know that the GBC density is related to instantons for Einstein space-time.

V. TOPOLOGICAL STRUCTURE OF GBC DENSITY ON THE MANIFOLD WITH BOUNDARY

When we take a manifold $\mathbf{M}$ with boundary $\partial\mathbf{M}$, the Euler characteristic becomes

$$\chi = \int_{n(\mathbf{M})} d\Omega + \int_{n(\partial\mathbf{M})} \Omega \tag{63}$$

The discussion of the first integral term on the left-hand side of the above equation keeps the same as what we discussed in the manifold without boundary

$$\int_{n(\mathbf{M})} d\Omega = \sum_{i=1}^{l} \int_{\mathbf{M}} \beta_i \eta_i \delta(x - z_i) d^n x = \sum_{i=1}^{l} \beta_i \eta_i \tag{64}$$

Hence, what we need to discuss is the second integral term on the right-hand side of equation (63).

With the same reason as mentioned before, this integral terms is induced as

$$\int_{n(\partial\mathbf{M})} \Omega = \int_{\partial\mathbf{M}} \frac{1}{(n-1)! A(S^{N-1})} \epsilon_{a_1 a_2 \cdots a_N} n_{a_1} dn_{a_2} \wedge \cdots \wedge dn_{a_N}. \tag{65}$$

Denote $m_\perp^a$ as the unit vector orthonormal to the boundary of $\mathbf{M}$, i.e.

$$m_\perp^a \perp \partial\mathbf{M} \tag{66}$$

and $m_{//}^a$ be the unit vector which is parallel to $\partial\mathbf{M}$. Now n can be represented by $m_\perp^a$ and $m_{//}^a$

$$n^a = m_\perp^a \cos\alpha + m_{//}^a \sin\alpha. \tag{67}$$

The projection of ϕ on the boundary $\partial\mathbf{M}$ is denoted as $\phi_{//}$

$$\phi_{//}^a = \phi^a - m_\perp^a \phi^b m_\perp^b \tag{68}$$

By choosing the direction of the basis of the vector field, we can make $m_\perp$ taking as $(0, \cdots 0, 1)$ then it can proved $m_{//}$ takes the form

$$m_{//} = (m^1, \cdots, m^{N-1}, 0) \tag{69}$$

and

$$m^A m^A = 1 \qquad\qquad A = 1, 2, \cdots, N-1 \tag{70}$$

Then we have

$$n = (m^A \sin\alpha, \cos\alpha) \tag{71}$$

and

$$\phi_m = (\phi_{//}^1, \cdots, \phi_{//}^{N-1}, 0) \tag{72}$$

Using this formula we can in forth to express the Chern-density as

$$\begin{aligned}\Omega = &\frac{1}{(n-1)! A(S^{N-1})} \epsilon_{A_1 A_2 \cdots A_N} (\sin^{N-1}\alpha \cos\alpha dm^{A_1} \wedge \cdots \wedge dm^{A_{N-1}} \\ &+ (N-1) m^{A_1} \sin^{N-2}\alpha d\alpha \wedge dm^{A_2} \wedge \cdots \wedge dm^{A_{N-1}})\end{aligned} \tag{73}$$

which can be written as the sum of two parts

$$\Omega = \rho_1 + \rho_2 \tag{74}$$

and

$$\rho_1 = \frac{\epsilon_{A_1 A_2 \cdots A_N}}{(N-1)! A(S^{N-1})} \sin^{N-1}\alpha \cos\alpha dm^{A_1} \wedge \cdots \wedge dm^{A_{N-1}} \tag{75}$$

$$\rho_2 = \frac{\epsilon_{A_1 A_2 \cdots A_N}}{(N-1)! A(S^{N-1})} m^{A_1} \sin^{N-2}\alpha d\alpha \wedge dm^{A_2} \wedge \cdots \wedge dm^{A_{N-1}} \tag{76}$$

By choice a term of coordinate $(u^1, \cdots, u^{N-1}, v)$ on the boundary $\partial\mathbf{M}$ and let $u = (u^1, \cdots, u^{N-1})$ be the inner coordinate of $\partial\mathbf{M}$. Using the same method in discussing the Chern density, we can get ρ_1 to be

$$\rho_1 = \frac{1}{(n-1)! A(S^{N-1})} \sin^{N-1}\alpha \cos\alpha \delta^{N-1}(\phi_{//}) d^{N-1}u \tag{77}$$

Notice that only at the zeroes of $\phi_{//}$ the delta function does not vanish. However at the zeroes of $\phi_{//}$ the angle α is 0 or π. Then

$$\sin^{N-1}\alpha \cos\alpha = 0 \tag{78}$$

Hence, ρ_1 contribute nothing to the Euler number for at the zeroes of $\phi_{//}$.

Now we turn to study the properties of ρ_2. For the continuous of $\phi_{//}$, there must exist a closed $N-2$-dimensional hypersurface $S^{N-2}(\alpha)$ in $\partial\mathbf{M}$, on which the angle α keeps the same. Therefore we can divide the integral of ρ_2 as

$$\begin{aligned}\int_{\partial\mathbf{M}} \rho_2 &= \frac{\epsilon_{A_1 \cdots A_{N-1}}}{2(N-2)! A(S^{N-1})} \int_0^\pi \sin^{N-2}\alpha d\alpha \int_{s(\alpha)} m^{A_1} dm^{A_2} \wedge \cdots \wedge dm^{A_{N-1}} \\ &= \frac{\epsilon_{A_1 \cdots A_{N-1}}}{2(N-2)! A(S^{N-2})} \int_{s(\alpha)} m^{A_1} dm^{A_2} \wedge \cdots \wedge dm^{A_{N-1}}\end{aligned} \tag{79}$$

in which we use the integral formula

$$\int_0^\pi \sin^{N-2}\alpha d\alpha = \sqrt{\pi} \frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N}{2})} \tag{80}$$

and the factor $\frac{1}{2}$ comes from that the integral limits $\alpha = 0$ and $\alpha = \pi$ should be seen as one points, which is computed twice. Therefore we must insert the factor $\frac{1}{2}$ to make the compute correct. Denote the $N-1$ dimensional super surface enclosed by $S^{N-2}(\alpha)$ as $S^{N-1}(\alpha)$ then from the stoke's theorem we have

$$\int_{\partial\mathbf{M}} \rho_2 = \frac{1}{2(N-2)! A(S^{N-2})} \epsilon_{A_1 \cdots A_{N-1}} d\alpha \int_{s(\alpha)} dm^{A_1} \wedge dm^{A_2} \wedge \cdots \wedge dm^{A_{N-1}} \tag{81}$$

Now it easy to proved analogous to that was proved before that

$$\begin{aligned}\int_{\partial\mathbf{M}} \rho_2 &= \frac{1}{2} \int_{\partial\mathbf{M}} \delta(\phi_{//}) J(\frac{\phi_{//}}{u}) d^{N-1}u \\ &= \frac{1}{2} \sum_{l=1}^{k} \beta_l(\phi_{//}) \eta_l(\phi_{//}) = \frac{1}{2} \deg \phi_{//}\end{aligned} \tag{82}$$

where k is the number of the zeroes of $\phi_{//}$ and $\beta_l(\phi_{//})$, $\eta_l(\phi_{//})$ are the Brouwer degree and Hopf index of $\phi_{//}$ on the l-th zero of $\phi_{//}$. Finally we obtain the index theorem of the manifold $\mathbf{M}$ with boundary $\partial\mathbf{M}$

$$\chi(\mathbf{M}) = \sum_{i=1}^{l} \beta_i(\phi)\eta_i(\phi) + \frac{1}{2}\sum_{i=1}^{k} \beta_i(\phi_{//})\eta_i(\phi_{//}). \tag{83}$$

In terms of winding umbers and indices, it is

$$\begin{aligned} \chi(\mathbf{M}) &= \sum_{i=1}^{l} W_i(\phi) + \frac{1}{2}\sum_{i=1}^{k} W_i(\phi_{//}) \\ &= \deg\phi + \frac{1}{2}\deg\phi_{//} \end{aligned} \tag{84}$$

This formula give a revise of the Hopf theorem on the manifold with boundary. If the angle α keeps invariant, especially when $\alpha = 0$, i.e. ϕ is parallel to $\partial\mathbf{M}$, One can find that the boundary term vanishes, and

$$\chi(\mathbf{M}) = \sum_{i=1}^{l} W_i(\phi) = \deg\phi. \tag{85}$$

VI. CONCLUSION

In conclusion, we have explicitly constructed the general decomposition theory of the spin connection of the group $SO(N)$ by virtue of N orthonormal vectors on the compact and oriented Riemannian manifold via Clifford algebra. We have proved the Euler-Poincaré characteristic $\chi(\mathbf{M})$ of a manifold $\mathbf{M}$ with boundary $\partial\mathbf{M}$ is equal to the sum of the total index of a smooth vector field ϕ on $\mathbf{M}$ and half the total index of the projective vector field of ϕ on $\partial\mathbf{M}$. The boundary term vanishes when the vector field is always transversal to the boundary and pointed outwards. Our results indicate that the Hopf indices and Brouwer degrees label the local structure of the Euler density. The Euler-Poincaré characteristic relates to the index, or equivalently the winding number, of the vector filed ϕ.

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